ACADEMIC PRESS

J. Math. Anal. Appl. 269 (2002) 689–701

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

# On the index integral transformation with Nicholson's function as the kernel

Semyon B. Yakubovich

*Department of Pure Mathematics, Faculty of Sciences, University of Porto, 4099-002 Porto, Portugal*

Received 26 September 2000; received in revised form 12 September 2001;
accepted 30 November 2001

Submitted by H.M. Srivastava

**Abstract**

The integral transformation, which is associated with the Nicholson function as the kernel, is introduced and investigated in the paper. This transformation is an integral, where integration is with respect to an index of the sum of squares of Bessel functions of the first and second kind. Composition representations and relationships with the Meijer $K$-transform, the Kontorovich–Lebedev transform, the Mellin transform, and the sine Fourier transform are given. We also present boundedness properties, a Parseval type equality, and an inversion formula. © 2002 Elsevier Science (USA). All rights reserved.

*Keywords:* Nicholson function; Bessel functions; Meijer transform; Sine Fourier transform; Mellin transform; Kontorovich–Lebedev transform; Parseval identity

## 1. Introduction and preliminary results

Let $x \in \mathbf{R}_+$ and consider the following integral transformation of the non-convolution type:

$$g(x) = \frac{\pi}{4} \frac{d}{dx} \int_0^\infty \left[J^2_{i\tau/2}(\sqrt{x}) + Y^2_{i\tau/2}(\sqrt{x})\right] \tau f(\tau)\, d\tau, \tag{1.1}$$

*E-mail address:* syakubov@fc.up.pt (S.B. Yakubovich).

0022-247X/02/$ – see front matter 
PII: S0022-247X(02)00043-4

where the kernel is given by the Nicholson function [1, p. 54], which is the sum of squares of Bessel functions of the first and second kind and which has the following integral representation:

$$J^2_{i\tau/2}\bigl(\sqrt{x}\bigr) + Y^2_{i\tau/2}\bigl(\sqrt{x}\bigr) = \frac{8}{\pi^2}\int_0^\infty K_0\bigl(2\sqrt{x}\sinh t\bigr)\cos(\tau t)\,dt. \tag{1.2}$$

Here $K_0(z)$ is the Macdonald function of index zero. The derivative in (1.1) exists for almost all $x > 0$ and the integration is with respect to the index of Bessel functions.

Our main goal in this paper is to consider the transformation in (1.1) in suitable functional spaces, to study its composition properties with other familiar integral transformations, and to prove an inversion theorem. Related problems were investigated in [2,3] for index transforms of Titchmarsh type with a linear combination of Bessel functions as kernels. The obtained results complete the theory and list of pairs of integral transformations of index type (cf. [4,5]), for which the kernels are special functions and integration is with respect to a parameter or the variable of these special functions.

As we will see below, the transform (1.1) is a particular case of the special class of integral operators of the form (cf. [6])

$$(\mathcal{G}_{\mathcal{H}} f)(x) = \int_0^\infty H(x,\tau)\tau f(\tau)\,d\tau, \tag{1.3}$$

where the kernel $H$ is given as a Mellin integral [7]; i.e.,

$$H(x,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{H}(s,\tau)x^{-s}\,ds, \quad s = \gamma + it,\ x \geqslant 0. \tag{1.4}$$

Note that if $H(x,\tau) = k(x\tau)$ then $\mathcal{H}(s,\tau) = k^{\mathcal{M}}(s)\tau^{-s}$, where $k^{\mathcal{M}}(s)$ is the Mellin transform of $k$:

$$k^{\mathcal{M}}(s) = \int_0^\infty k(x)x^{s-1}\,dx.$$

Such operators are called Mellin convolution type operators or general transformations of Fourier type. They were investigated in [7, Chapter VIII]. It is natural to use $L_2$-Mellin transform theory to obtain boundedness properties and inversion properties for such operators. However, in our case the kernel $H(x,\tau)$ is essentially a function of two variables and to construct the theory of such a transform we will use its relationship with the Kontorovich–Lebedev transformation (see [4]).

Indeed, if we look at the table of the Mellin transforms for hypergeometric functions in [8], most examples there contain $\tau$ as a parameter in gamma functions and they lead to integral transform operators of non-convolution type. Among them one can find the Kontorovich–Lebedev, the Mehler–Fock, the Olevskii and other transformations. An example is

$$\frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)x^{-s}\,ds = K_{i\tau}\big(2\sqrt{x}\big), \quad \gamma>0, \tag{1.5}$$

where $\mathcal{H}(s,\tau)$ is equal to the product of two gamma functions. It defines the Macdonald function of a pure imaginary index and the following Kontorovich–Lebedev transform:

$$[KL\, f](x) = 2\int_0^\infty \tau K_{i\tau}\big(2\sqrt{x}\big) f(\tau)\,d\tau. \tag{1.6}$$

The relation (8.4.41.10) in [8], under the condition $0<\gamma<1/2-\mu$, gives the following Mellin–Barnes integral representation for the generalized Legendre function [1]:

$$\begin{aligned}&\big|\Gamma\big((1+i\tau)/2-\mu\big)\big|^2 x^{-1/2}(1+x)^{\mu/2}P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)\\ &\quad=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\frac{\Gamma(1/2-\mu-s)}{\Gamma(1/2+s)}x^{-s}\,ds.\end{aligned} \tag{1.7}$$

The corresponding index transform is the generalized Mehler–Fock operator [4]

$$\begin{aligned}[MF\, f](x) = x^{-1/2}(1+x)^{\mu/2}&\int_0^\infty \tau\big|\Gamma\big((1+i\tau)/2-\mu\big)\big|^2\\ &\times P^{\mu}_{(i\tau-1)/2}\left(\frac{2}{x}+1\right)f(\tau)\,d\tau.\end{aligned} \tag{1.8}$$

The classical Mehler–Fock transform is the one with $\mu=0$.

The Mellin–Barnes integral representation for the Nicholson function is given by formula (8.4.20.35) from [8],

$$\begin{aligned}&\frac{\pi^{5/2}}{2\cosh(\pi\tau/2)}\big[J^2_{i\tau/2}\big(\sqrt{x}\big)+Y^2_{i\tau/2}\big(\sqrt{x}\big)\big]\\ &\quad=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\Gamma(s)\Gamma(1/2-s)x^{-s}\,ds,\end{aligned} \tag{1.9}$$

where $0 < \gamma < 1/2$. Differentiating with respect to $x$ under the integral sign in (1.9) (this can be done because of uniform convergence) and taking into account the recurrence formula for the Euler gamma function gives another useful representation

$$-\frac{\pi^{5/2}}{2\cosh(\pi\tau/2)} x \frac{d}{dx}\left[J^2_{i\tau/2}(\sqrt{x}) + Y^2_{i\tau/2}(\sqrt{x})\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(s+\frac{i\tau}{2}\right)\Gamma\left(s-\frac{i\tau}{2}\right)\Gamma(1+s)\Gamma(1/2-s)x^{-s}\,ds. \tag{1.10}$$

In order to obtain the inversion of the Nicholson transformation (1.1) and to prove the Parseval type equality, we give one more Mellin–Barnes integral representation for the combination of squares of Bessel functions. Using formula (8.4.19.19) in [8], we have

$$\frac{\sqrt{\pi}}{2i\sinh(\pi\tau/2)}\left[J^2_{-i\tau/2}(\sqrt{x}) - J^2_{i\tau/2}(\sqrt{x})\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+i\tau/2)\Gamma(s-i\tau/2)}{\Gamma(1/2+s)\Gamma(1-s)} x^{-s}\,ds, \tag{1.11}$$

where $0 < \gamma < 1/4$.

Finally, we give the Parseval equality for the Kontorovich–Lebedev transform (1.6) (cf. also [4,9,10]), which will be useful later:

$$\int_0^\infty \left|[KL\, f](x)\right|^2 \frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} |f(\tau)|^2\, d\tau. \tag{1.12}$$

Note that this transformation is an isometric isomorphism between the Hilbert spaces $L_2(\mathbf{R}_+; \tau\, d\tau/\sinh(\pi\tau))$ and $L_2(\mathbf{R}_+; x^{-1}\, dx)$. Moreover, the inverse of the Kontorovich–Lebedev transform is given by the formula

$$f(\tau) = \frac{1}{2\pi^2} \lim_{N\to\infty} \sinh(\pi\tau) \int_{1/N}^{N} K_{i\tau}\left(2\sqrt{x}\right)[KL\, f](x)\frac{dx}{x}, \tag{1.13}$$

where the convergence of the latter integral is in the norm of $L_2(\mathbf{R}_+; \tau\, d\tau/\sinh(\pi\tau))$.

## 2. Composition representation of the Nicholson transform

In this section we will establish a composition representation of the transform (1.1) through the Meijer $K$-transform [11] and the sine Fourier transform

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \sin(\tau x)\, d\tau. \tag{2.1}$$

As it is known from [1], the Macdonald function $\rho(x,t) = 2K_0(2\sqrt{x}\sinh t)$ in (1.2) satisfies the following ordinary differential equation:

$$\frac{\partial}{\partial x}\left(x \frac{\partial}{\partial x}\rho(x,t)\right) = \rho(x,t)\sinh^2 t. \tag{2.2}$$

Assume that $f \in C_0^\infty(\mathbf{R}_+)$, the space of smooth functions with compact support, substitute the integral (1.2) into (1.1) and change the order of integration. This is possible in view of the Fubini–Tonelli theorem and the asymptotic behavior of the Macdonald function (cf. [1]):

$$K_0(x) = O(\log x), \quad x \to 0+, \tag{2.3}$$

$$K_0(x) = O\big(e^{-x}/\sqrt{x}\big), \quad x \to +\infty. \tag{2.4}$$

Then, as it is not difficult to see, we find

$$g(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty K_0\big(2\sqrt{x}\sinh t\big) \frac{d}{dt}\big[(F_s f)(t)\big]\, dt. \tag{2.5}$$

The latter integral may be integrated by parts. Since the integrated terms vanish and via the differential property of the Macdonald function

$$\frac{d}{dx} K_0(x) = -K_1(x), \tag{2.6}$$

where $K_1(x)$ is the Macdonald function of index 1, we will arrive at the following equality:

$$g(x) = 2\sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty \sqrt{x} K_1\big(2\sqrt{x}\sinh t\big)(F_s f)(t)\cosh t\, dt. \tag{2.7}$$

Hence, because to the uniform convergence of the corresponding integral, we can differentiate with respect to $x$ in (2.7) and then apply the differential equation (2.2) to the obtained kernel function. As a result we obtain

$$g(x) = -2\sqrt{\frac{2}{\pi}} \int_0^\infty K_0\big(2\sqrt{x}\sinh t\big)(F_s f)(t)\cosh t \sinh t\, dt. \tag{2.8}$$

Hence, substituting $\sinh^2 t = u$ in (2.8) we find the following composition representation of the Nicholson transform (1.1)

$$g(x) = -\sqrt{\frac{2}{\pi}} \int_0^\infty K_0\bigl(2\sqrt{xu}\bigr)(F_s f)\bigl(\sinh^{-1}\bigl(\sqrt{u}\bigr)\bigr)\,du \tag{2.9}$$

in terms of the sine Fourier transform (2.1) evaluated at the point $\sinh^{-1}(\sqrt{u}) = \log(\sqrt{u} + \sqrt{1+u})$ and the Meijer transform of the Mellin convolution type with the Macdonald function $K_0(2\sqrt{xu})$ as the kernel.

## 3. An analog of the Parseval identity for the Nicholson transform

Let us consider the following modified Nicholson transformation, which is a particular case of the general transform (1.3):

$$[JYf](x) = -\frac{\pi^{5/2}}{2} x \int_0^\infty \frac{d}{dx}\bigl[J^2_{i\tau/2}\bigl(\sqrt{x}\bigr) + Y^2_{i\tau/2}\bigl(\sqrt{x}\bigr)\bigr]\frac{\tau f(\tau)}{\cosh(\pi\tau/2)}\,d\tau. \tag{3.1}$$

For 'sufficiently good' functions from $C_0^\infty(\mathbf{R}_+)$ one can use the asymptotic behavior of the kernel and the uniform convergence to write the integral (3.1) in the form

$$[JYf](x) = -\frac{\pi^{5/2}}{2} x \frac{d}{dx} \int_0^\infty \bigl[J^2_{i\tau/2}\bigl(\sqrt{x}\bigr) + Y^2_{i\tau/2}\bigl(\sqrt{x}\bigr)\bigr]\frac{\tau f(\tau)}{\cosh(\pi\tau/2)}\,d\tau. \tag{3.2}$$

In a similar way we introduce the integral operator with the kernel (1.11) as

$$[J^2 f](x) = \frac{\sqrt{\pi}}{2i} \int_0^\infty \bigl[J^2_{-i\tau/2}\bigl(\sqrt{x}\bigr) - J^2_{i\tau/2}\bigl(\sqrt{x}\bigr)\bigr]\frac{\tau f(\tau)}{\sinh(\pi\tau/2)}\,d\tau. \tag{3.3}$$

In this section we will prove that both operators (3.1) and (3.3) are bounded in weighted Lebesgue $L_2$-spaces and satisfy the following Parseval type equality:

$$\int_0^\infty [JYf](x)\overline{[J^2 g](x)}\,\frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} f(\tau)\overline{g(\tau)}\,d\tau. \tag{3.4}$$

In particular, when $f = g$, $f \in L_2(\mathbf{R}_+)$ we will get the equality (3.4) in the form

$$\int_0^\infty [JYf](x)\overline{[J^2 f](x)}\,\frac{dx}{x} = 4\pi^2 \int_0^\infty \frac{\tau}{\sinh(\pi\tau)} |f(\tau)|^2\,d\tau. \tag{3.5}$$

**Theorem 1.** *The Nicholson transformation*

$$[JY\, f] : L_2\left(\mathbf{R}_+; \frac{\tau\, d\tau}{\sinh(\pi\tau)}\right) \to L_2(\mathbf{R}_+; x^{-1}\, dx), \tag{3.6}$$

*is a bounded operator given by the formula* (3.1)*, where the integral* (3.1) *exists as a Lebesgue integral.*

*Moreover, it satisfies the Parseval type equality* (3.4)*, where the auxiliary operator*

$$[J^2 g] : L_2(\mathbf{R}_+) \to L_2(\mathbf{R}_+; x^{-1}\, dx) \tag{3.7}$$

*is bounded and can be defined by the formula* (3.3) *with the convergence of the corresponding integral in the mean square sense for the norm of the space* $L_2(\mathbf{R}_+; x^{-1}\, dx)$*. In particular, on the subspace* $f \in L_2(\mathbf{R}_+) \subset L_2(\mathbf{R}_+; \tau\, d\tau/\sinh(\pi\tau))$ *the identity* (3.5) *holds true.*

**Sketch of the proof.** For the time being we assume that $f, g \in C_0^\infty(\mathbf{R}_+)$. If we denote

$$\Theta_f(s) = \int_0^\infty \tau f(\tau) \Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) d\tau \tag{3.8}$$

and substitute the integral representations (1.10) and (1.11) of the corresponding kernels into (3.2) and (3.3), then after changing the order of integration we obtain

$$[JY\, f](x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Theta_f(s) \varphi(s) x^{-s}\, ds, \tag{3.9}$$

$$[J^2 g](x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Theta_g(s) \psi(s) x^{-s}\, ds, \tag{3.10}$$

where $\varphi(s) = \Gamma(1/2 - s)\Gamma(1 + s)$, $\psi(s) = [\Gamma(1/2 + s)\Gamma(1 - s)]^{-1}$, and $\gamma \in (0, 1/2)$.

Our goal now is to apply the $L_2$-theory of the Mellin transform (cf. [7]) in order to prove the following Parseval type equality

$$\int_0^\infty [JY\, f](x) \overline{[J^2 g](x)} x^{2\gamma - 1}\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^\infty [JY\, f]^{\mathcal{M}}(\gamma + it) \overline{[J^2 g]^{\mathcal{M}}(\gamma + it)}\, dt, \tag{3.11}$$

where the symbols $[JYf]^{\mathcal{M}}(\gamma+it)$ and $[J^2g]^{\mathcal{M}}(\gamma+it)$ denote compositions of the Mellin transform with the operators in (3.9) and (3.10), evaluated at the point $\gamma+it$. We will establish that operators $JYf$, $J^2g$ are bounded and their images belong to the space $L_2(\mathbf{R};x^{-1}\,dx)$. Moreover, from the asymptotic behaviour of Bessel functions one can show that $[JYf](x),[J^2g](x)\in L_2(\mathbf{R};x^{2\gamma-1}\,dx)$, $\gamma\in(0,1/2)$. This immediately implies the equality (3.11). Furthermore, in view of analytic properties of functions $\Theta_f(s),\Theta_g(s)$ in the strip $\gamma\in(0,1/2)$ the well-known Paley–Wiener theorem says that when $\gamma\to 0$, then functions $[JYf]^{\mathcal{M}}(\gamma+it),[J^2g]^{\mathcal{M}}(\gamma+it)$ have boundary values in $L_2$-sense. Consequently, one can put $\gamma=0$ in (3.11). Hence, via relations (3.9) and (3.10) we obtain $[JYf]^{\mathcal{M}}=\Theta_f(it)\varphi(it)$ and $[J^2g]^{\mathcal{M}}=\Theta_g(it)\psi(it)$. Substituting these values into (3.11) we obtain

$$\int_0^\infty [JYf](x)\overline{[J^2g](x)}\,\frac{dx}{x}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta_f(it)\overline{\Theta_g(it)}\varphi(it)\overline{\psi(it)}\,dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta_f(it)\overline{\Theta_g(it)}\,dt, \tag{3.12}$$

where, as it is easily seen, $\varphi(it)\overline{\psi(it)}=1$. With the Schwarz inequality the left-hand side of (3.12) can be estimated as

$$\int_0^\infty [JYf](x)\overline{[J^2g](x)}\,\frac{dx}{x}\leqslant \big\|[JYf]\big\|_{L_2(\mathbf{R}_+;x^{-1}\,dx)}\big\|[J^2g]\big\|_{L_2(\mathbf{R}_+;x^{-1}\,dx)}. \tag{3.13}$$

By using the integral (1.5) for $\gamma>0$, substitute it in (1.6) and change the order of integration. Then as a result we obtain that the composition of the Mellin and Kontorovich–Lebedev transforms $[KLf]^{\mathcal{M}}(\gamma+it)=\Theta_f(\gamma+it)$, $\gamma>0$. One can prove (see, e.g., [6]) that $[KLf](x)\in L_2(\mathbf{R}_+;x^{-1}\,dx)$ for each $f\in C_0^\infty(\mathbf{R}_+)$, so that $\Theta_f(it)$ is defined.

Returning to the equality (3.12) and combining it with (1.12), we continue its right-hand side as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Theta_f(it)\overline{\Theta_g(it)}\,dt=\int_0^\infty [KLf](x)\overline{[KLg](x)}\,\frac{dx}{x}$$
$$=4\pi^2\int_0^\infty \frac{\tau}{\sinh(\pi\tau)}f(\tau)\overline{g(\tau)}\,d\tau. \tag{3.14}$$

Thus the equality (3.12) turns into the Parseval type equality (3.4). Moreover, it holds at least for $f\in C_0^\infty(\mathbf{R}_+)$. Hence this equality can be extended to functions from the Hilbert space where both operators (3.1) and (3.3) are bounded. In order to verify the boundedness of $[JY\,f]$ as an operator acting from $L_2(\mathbf{R}_+;\tau\,d\tau/\sinh(\pi\tau)) \to L_2(\mathbf{R}_+;x^{-1}\,dx)$, we apply its relationship with the Mellin transform given by (3.9), the Mellin–Parseval equality, and the Parseval equality for the Kontorovich–Lebedev transform (1.12). Indeed, we have

$$\begin{aligned}
\big\|[JY\,f]\big\|_{L_2(\mathbf{R}_+;x^{-1}\,dx)} &= \left(\int_0^\infty \big|[JY\,f](x)\big|^2 \frac{dx}{x}\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^\infty |\Theta_f(it)|^2|\varphi(it)|^2\,dt\right)^{1/2} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^\infty |\Theta_f(it)|^2|\Gamma(1/2-it)|^2|\Gamma(1+it)|^2\,dt\right)^{1/2} \\
&= \left(\pi\int_{-\infty}^\infty |\Theta_f(it)|^2\,\frac{t\,dt}{\sinh(2\pi t)}\right)^{1/2} \\
&\leqslant \sqrt{\pi}\left(\frac{1}{2\pi}\int_{-\infty}^\infty |\Theta_f(it)|^2\,dt\right)^{1/2} = \sqrt{\pi}\left(\int_0^\infty \big|[KL\,f](x)\big|^2\frac{dx}{x}\right)^{1/2} \\
&= 2\pi\sqrt{\pi}\left(\int_0^\infty \frac{\tau}{\sinh(\pi\tau)}|f(\tau)|^2\,d\tau\right)^{1/2} \\
&= 2\pi\sqrt{\pi}\,\|f\|_{L_2(\mathbf{R}_+;\tau\,d\tau/\sinh(\pi\tau))}.
\end{aligned} \tag{3.15}$$

In order to prove the boundedness of the auxiliary operator $[J^2 f]$ in the corresponding space of functions we use the fact (cf. [6]) that for any $f \in C_0^\infty(\mathbf{R}_+)$ the Mellin transform $[J^2 f]^{\mathcal{M}}(it)$ can be written in the form

$$[J^2 f]^{\mathcal{M}}(it) = 2\sqrt{\pi}\,\frac{\Gamma(1+it)}{\Gamma(1-it)}(\Phi f)(t), \tag{3.16}$$

with

$$(\Phi f)(x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i\xi x}(F_s f)(\operatorname{arccosh} e^{\xi/2})\,d\xi, \tag{3.17}$$

where $F_s$ is the sine Fourier transform (2.1).

Now we find

$$\left(\int_0^\infty \big|[J^2 f](x)\big|^2\frac{dx}{x}\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\infty}^\infty \big|\Theta_f(it)\psi(it)\big|^2\,dt\right)^{1/2}$$

$$= \left( 2 \int_{-\infty}^{\infty} |\Phi(t)|^2 \left| \frac{\Gamma(1+it)}{\Gamma(1-it)} \right|^2 dt \right)^{1/2} = \sqrt{2} \left( \int_{-\infty}^{\infty} |\Phi(t)|^2 \, dt \right)^{1/2}$$
$$= \sqrt{2} \left( \int_0^{\infty} \left| (F_s f)(\operatorname{arccosh} e^{\xi/2}) \right|^2 d\xi \right)^{1/2}$$
$$= 2 \left( \int_0^{\infty} \left| (F_s f)(u) \right|^2 \tanh u \, du \right)^{1/2} \leqslant 2 \left( \int_0^{\infty} \left| (F_s f)(u) \right|^2 du \right)^{1/2}$$
$$= 2 \left( \int_0^{\infty} |f(y)|^2 \, dy \right)^{1/2} = 2 \|f\|_{L_2(\mathbf{R}_+)}. \tag{3.18}$$

Since the space $C_0^\infty(\mathbf{R}_+)$ is dense in $L_2(\mathbf{R}_+)$, we may extend (3.18) to any $f \in L_2(\mathbf{R}_+)$ and obtain the boundedness of the operator $[J^2 f] : L_2(\mathbf{R}_+) \to L_2(\mathbf{R}_+; x^{-1} dx)$. Invoking the embedding $L_2(\mathbf{R}_+) \subset L_2(\mathbf{R}_+; \tau \, d\tau / \sinh(\pi\tau))$ we conclude that the Parseval equality (3.5) is valid for any $f \in L_2(\mathbf{R}_+)$.

Turning to the formula (3.1), we can now show that the corresponding integral exists for any $f \in L_2(\mathbf{R}_+; \tau \, d\tau / \sinh(\pi\tau))$. Indeed, first we use another integral representation for the Nicholson function which follows directly from (1.9) by using the integral (1.5), the Euler beta integral and the Mellin convolution theory from [7]. This gives the following integral:

$$\frac{1}{\cosh(\pi\tau/2)} \left[ J_{i\tau/2}^2(\sqrt{x}) + Y_{i\tau/2}^2(\sqrt{x}) \right]$$
$$= \frac{4}{\pi^2} \int_0^{\infty} \frac{K_{i\tau}(2\sqrt{v})}{\sqrt{v+x}} \frac{dv}{\sqrt{v}}, \quad x > 0. \tag{3.19}$$

Applying the differential operator $x(d/dx)$ to both sides of (3.19) and differentiating through the integral sign, we arrive at the following representation for the kernel function of the Nicholson transformation:

$$\frac{1}{\cosh(\pi\tau/2)} x \frac{d}{dx} \left[ J_{i\tau/2}^2(\sqrt{x}) + Y_{i\tau/2}^2(\sqrt{x}) \right]$$
$$= -\frac{2x}{\pi^2} \int_0^{\infty} \frac{K_{i\tau}(2\sqrt{v})}{(v+x)^{3/2}} \frac{dv}{\sqrt{v}}, \quad x > 0. \tag{3.20}$$

It is not difficult to verify that for each $x > 0$ we have $\sqrt{v}/(v+x)^{3/2} \in L_2(\mathbf{R}_+; v^{-1} dv)$. Moreover, via the inversion formula (1.13) for the Kontorovich–Lebedev transform, we immediately obtain that

$$\sinh(\pi\tau/2) \frac{d}{dx} \left[ J_{i\tau/2}^2(\sqrt{x}) + Y_{i\tau/2}^2(\sqrt{x}) \right] \in L_2\left( \mathbf{R}_+; \frac{\tau \, d\tau}{\sinh(\pi\tau)} \right).$$

Consequently, via the Schwarz inequality one can conclude that the integral (3.1) exists as a Lebesgue integral. Finally, one can verify the convergence of the integral (3.3) over the index of squares of Bessel functions, which can be written for all $x > 0$ in the form

$$[J^2 f](x) = -\sqrt{\pi} \int_0^\infty \operatorname{Im} J^2_{i\tau/2}\big(\sqrt{x}\big) \frac{\tau f(\tau)}{\sinh(\pi\tau/2)}\, d\tau, \tag{3.21}$$

where $\operatorname{Im} J^2_{i\tau/2}(\sqrt{x})$ is the imaginary part of the function $J^2_{i\tau/2}(\sqrt{x})$ and $f \in C_0^\infty(\mathbf{R}_+)$. Since the kernel of the integral transform (3.21) is a continuous function in $\tau$, the integral

$$[J^2 f]_N(x) = -\sqrt{\pi} \int_0^N \operatorname{Im} J^2_{i\tau/2}\big(\sqrt{x}\big) \frac{\tau f(\tau)}{\sinh(\pi\tau/2)}\, d\tau$$

exists for any $f \in L_2(\mathbf{R}_+)$. Furthermore, according to the chain of relations (3.18), we have

$$\begin{aligned} \big\| [J^2 f] - [J^2 f]_N \big\|^2_{L_2(\mathbf{R}_+; x^{-1}dx)} &\leqslant 2\| f - f_N \|^2_{L_2(\mathbf{R}_+)} \\ &= 2 \int_N^\infty |f(y)|^2\, dy \to 0, \end{aligned} \tag{3.22}$$

as $N \to \infty$, where $f_N(\tau) = f(\tau)$, $\tau \in [0, N]$ and $f_N = 0$, $\tau \in [N, \infty)$. Consequently, the integral (3.3) exists in the mean square sense. This completes the proof of Theorem 1. □

## 4. Inversion theorem for the Nicholson transform

The Parseval type formula (3.4) gives a key to prove the inversion theorem for the integral transform (3.1). Indeed, choosing

$$g(y) = \begin{cases} 1, & \text{if } y \in [0, \tau], \\ 0, & \text{if } y \in (\tau, \infty), \end{cases}$$

we arrive at the equality

$$\int_0^\infty [JYf](x) \overline{[J^2 g](x,\tau)}\, \frac{dx}{x} = 4\pi^2 \int_0^\tau \frac{y}{\sinh(\pi y)} f(y)\, dy, \tag{4.1}$$

where (cf. (3.3))

$$[J^2 g](x,\tau) = \frac{\sqrt{\pi}}{2i} \int_0^\tau \big[ J^2_{-iy/2}\big(\sqrt{x}\big) - J^2_{iy/2}\big(\sqrt{x}\big) \big] \frac{y}{\sinh(\pi y/2)}\, dy. \tag{4.2}$$

Hence, as a corollary from (4.1), we obtain for almost all $\tau \in \mathbf{R}_+$

$$f(\tau) = -\frac{1}{4\pi\sqrt{\pi}} \frac{\sinh(\pi\tau)}{\tau} \frac{d}{d\tau} \int_0^\infty [JY f](x) \times \int_0^\tau \operatorname{Im} J_{iy/2}^2(\sqrt{x}) \frac{y}{\sinh(\pi y/2)} \, dy \, \frac{dx}{x}. \tag{4.3}$$

From the asymptotic behaviour of Bessel functions one can prove that

$$\int_0^\tau \operatorname{Im} J_{iy/2}^2(\sqrt{x}) \frac{y}{\sinh(\pi y/2)} \, dy \in L_2(\mathbf{R}_+; x^{-1}\, dx). \tag{4.4}$$

This would imply that the integral (4.3) exists as a Lebesgue integral. Thus we obtained the inversion formula for the Nicholson transformation (3.1). It can be written in $L_2$-convergence of integrals by passing the derivative through the integral:

$$f(\tau) = -\frac{1}{2\pi\sqrt{\pi}} \lim_{N\to\infty} \cosh\left(\frac{\pi\tau}{2}\right) \int_{1/N}^N \operatorname{Im} J_{i\tau/2}^2(\sqrt{x}) [JY f](x) \frac{dx}{x}. \tag{4.5}$$

**Theorem 2.** *For any $f \in L_2(\mathbf{R}_+; \tau\, d\tau/\sinh(\pi\tau))$ the Nicholson transformation* (3.1) *exists for almost all $x > 0$ as a bounded operator and belongs to $L_2(\mathbf{R}_+; x^{-1}\, dx)$. There is a one-to-one correspondence with the inversion formula* (4.5), *where the limit is the mean square sense for the norm in $L_2(\mathbf{R}_+; \tau\, d\tau/\sinh(\pi\tau))$.*

## Acknowledgments

This work was supported in part by the "Centro de Matemática" of the University of Porto. The author sincerely thanks Walter Van Assche for valued comments of this work and useful suggestions.

## References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, Vol. II, McGraw–Hill, New York, 1953.

[2] S.B. Yakubovich, Index integral transformation of Titchmarsh type, J. Comput. Appl. Math. 85 (1997) 169–179.

[3] S.B. Yakubovich, The Titchmarsh integral transformation by the index of a Bessel function, J. Comput. Appl. Math. (Special Issue) 118 (2000) 353–361.

[4] S.B. Yakubovich, Yu.F. Luchko, The Hypergeometric Approach to Integral Transforms and Convolutions, Math. Appl., Vol. 287, Kluwer Academic, Dordrecht, 1994.

[5] S.B. Yakubovich, B. Fisher, A class of index transforms with general kernels, Math. Nachr. 200 (1999) 165–182.
[6] S.B. Yakubovich, J. de Graaf, On Parseval equalities and boundedness properties for Kontorovich–Lebedev type operators, Novi Sad J. Math. 29 (1999) 185–205.
[7] E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Clarendon Press, Oxford, 1937.
[8] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series: More Special Functions, Gordon and Breach, New York, 1989.
[9] J.S. Lowndes, Parseval relations for Kontorovich–Lebedev transform, Proc. Edinburgh Math. Soc. (2) 13 (1962) 5–11.
[10] S.B. Yakubovich, Index Transforms, World Scientific, Singapore, 1996.
[11] C.S. Meijer, Über eine Erweiterung der Laplace-Transformation, Nederl. Acad. Wetensch. Proc. 43 (1940) 599–608, 702–711.